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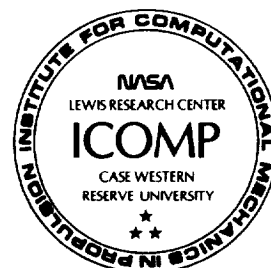
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## AN ALGORITHM FOR UNSTEADY FLOWS WITH STRONG CONVECTION

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### SUMMARY

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An implicit ADI numerical method for the calculation of two-dimensional unsteady flows with strong convection effects is described. The method is based upon the conventional Crank-Nicholson approach for parabolic equations but an upwind-downwind differencing is used for the first order spatial derivatives associated with convection. The differencing is carried out in the current and previous time plane in such a way that the algorithm is second order accurate in both space and time. The difference equations are factored into sequential operators, one in each independent spatial variable; the solution at each time step may then be computed as a sequence of tridiagonal matrix problems. The method may be used in a noniterative manner although iteration at each time step is recommended in situations where the effects of convection are strong.

### INTRODUCTION

Unsteady flows with strong convection effects occur in a variety of circumstances. Many of these situations are associated with unsteady boundary-layer separation phenomena and the strong unsteady viscous-inviscid interactions that are observed to occur between an outer effectively inviscid flow and the viscous flow near a solid surface. Specific examples include (i) small separation bubbles on the upper surface of turbine blades and airfoils which in certain situations erupt into the inviscid flow region, (ii) the eruption of boundary-layer flows which is induced by the motion of vortices near solid walls (refs. 1 and 2) and (iii) bursting in turbulent boundary layers (refs. 3 and 4). One feature of the aforementioned flow situations is that, as the interaction initiates, very strong updrafts begin to develop in the boundary layers near the wall and the flow field locally is dominated by strong convection effects.

When finite difference schemes are used to compute the evolution of a time-dependent flow, methods based on some version of either the Crank-Nicholson algorithm (ref. 1) or a factored-operator ADI (Alternating-Direction-Implicit) technique are used. In Crank-Nicholson methods, the difference equations in the current time plane are usually solved by iteratively sweeping point-by-point through the two-dimensional spatial mesh

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until convergence is attained at each point. In the factored alternating-direction methods (ref. 5), the difference operators are factored and the difference equations are solved as an ordered sequence of tridiagonal matrix problems in each of the coordinate directions. In general, the ADI methods are more efficient than the iterative Crank-Nicolson method. However, in a recent study of the evolution of unsteady flow in a two-dimensional boundary layer (ref. 6), the explicit Beam and Warming (ref. 5) type of algorithm was found to fail in situations where strong time-dependent convective effects were present. The algorithm described in this study is also a factored ADI method; however, upwind-downwind differencing is used for first order convective derivatives and the method is also implicit. The present method produced good results (ref. 6) in a situation where both standard Crank-Nicolson method and conventional ADI methods failed.

### Problem Statement

The two-dimensional unsteady boundary-layer equations in either Eulerian or Lagrangian coordinates can be written in the following general form:

$$\frac{\partial u}{\partial t} = R \frac{\partial^2 u}{\partial \xi^2} + S \frac{\partial^2 u}{\partial \xi \partial \eta} + T \frac{\partial^2 u}{\partial \eta^2} + P \frac{\partial u}{\partial \eta} + Q \frac{\partial u}{\partial \xi} + Wu + \Gamma, \quad (1)$$

where  $u$  is the tangential velocity in the boundary layer,  $(\xi, \eta)$  are independent spatial coordinates and  $t$  is time. In the Lagrangian description of the boundary-layer motion,  $(\xi, \eta)$  are the Cartesian coordinates of a particle at the initiation of the motion; if  $x(\xi, \eta, t)$  denotes the streamwise position of a fluid particle at time  $t$ , equation (1) is supplemented with an equation

$$\frac{\partial x}{\partial t} = u, \quad (2)$$

and the functional coefficients in equation (1) (namely  $R, S, T, P, Q, W$ ) depend on  $x$  and  $u$  as well as  $(\xi, \eta, t)$ . Consequently the system described by equations (1) and (2) is nonlinear and equation (2) is simply a convenient representation of the momentum equation. Note that the coefficients  $R$  and  $T$  are such that

$$R \geq 0, \quad T \geq 0. \quad (3)$$

In the Eulerian description of unsteady two-dimensional boundary-layer flow (see, for example, ref. 1),

$$R = S = 0, \quad T = 1, \quad (4)$$

and equation (1) is supplemented with

$$\frac{\partial \psi}{\partial \eta} = u, \quad (5)$$

where  $\psi$  is the two-dimensional stream function; in this case, the functional coefficients in equation (1) are functions of  $u$  and  $\psi$  as well as  $(\xi, \eta, t)$ .

It is worthwhile to note that each momentum equation for the full two-dimensional Navier-Stokes equations is also in the general form of equation (1).

In order to compute the evolution of the flow field, it is necessary to develop algorithms which advance the solution of equation (1) one time step at a time. Consider a spatial mesh in the  $\xi\eta$  plane and adopt a convention where subscripts  $ij$  denote a quantity evaluated at a typical point in the mesh, with  $i$  and  $j$  representing typical locations in the  $\xi$  and  $\eta$  directions respectively. It is assumed that the solution is known at time  $t^*$  (the previous time plane) and the objective is to compute the solution at  $t = t^* + \Delta t$  (the current time plane), where  $\Delta t$  is the time step. All quantities evaluated in the previous time plane are assumed known and will be indicated by an asterisk; values of  $u$  in the current time plane are unknown and to be found.

### The Conventional Crank-Nicolson Method

The grid structure near a typical spatial mesh point is indicated schematically in figure 1. In the standard Crank-Nicolson method (ref. 1), equation (1) is approximated at a point midway between the current and previous time planes along a line connecting the typical point in mesh labelled  $ij$ . The coefficients in equation (1) are evaluated through a simple averaging procedure involving quantities in the current and previous time planes; using an overbar to denote quantities evaluated at the midpoint, a typical coefficient in equation (1) is evaluated according to

$$\bar{p}_{ij} = \frac{1}{2} (p_{ij} + p^*_{ij}), \quad (6)$$

for example. For derivatives in equation (1), an average is also carried out between the current and previous time planes and central difference approximations are subsequently used in both time planes. Let  $\delta_\xi$  and  $\delta_\eta$  be central difference operators in the  $\xi$  and  $\eta$  directions respectively with  $\mu_\xi$  and  $\mu_\eta$  being averaging operators in the corresponding directions (ref. 7) (which are defined in the Appendix). Then central difference approximations for the derivatives in equation (1) in the current time plane at the point  $ij$  may be written

$$\frac{\partial^2 u}{\partial \eta^2} = \frac{1}{h_\eta^2} \delta_\eta^2 u_{ij} + O(\Delta \eta^2) \approx \frac{u_{i,j+1} - 2u_{ij} + u_{i,j-1}}{h_\eta^2}, \quad (7a)$$

$$\frac{\partial^2 u}{\partial \xi^2} = \frac{1}{h_\xi^2} \delta_\xi^2 u_{ij} + O(\Delta \xi^2) \approx \frac{u_{i+1,j} - 2u_{ij} + u_{i-1,j}}{h_\xi^2}, \quad (7b)$$

$$\frac{\partial u}{\partial \eta} = \frac{1}{h_\eta} \mu_\eta \delta_\eta u_{ij} + O(\Delta \eta)^2 \approx \frac{u_{i,j+1} - u_{i,j-1}}{2h_\eta}, \quad (7c)$$

$$\frac{\partial u}{\partial \xi} = \frac{1}{h_\xi} \mu_\xi \delta_\xi u_{ij} + O(\Delta \xi)^2 \approx \frac{u_{i+1,j} - u_{i-1,j}}{2h_\xi}, \quad (7d)$$

$$\begin{aligned} \frac{\partial^2 u}{\partial \xi \partial \eta} &= \frac{1}{h_\xi h_\eta} \mu_\xi \delta_\xi \mu_\eta \delta_\eta u_{ij} + O(\Delta \xi \Delta \eta) \\ &\approx \frac{u_{i+1,j+1} - u_{i-1,j+1} - u_{i+1,j-1} + u_{i-1,j-1}}{4h_\xi h_\eta}, \end{aligned} \quad (7e)$$

where  $h_\xi$  and  $h_\eta$  are the mesh spacings in the  $\xi$  and  $\eta$  directions respectively. Corresponding formulae apply in the previous time plane except that the values of  $u_{ij}$  appearing on the right sides of equations (7) are evaluated at  $t^*$  and would therefore be written with asterisks.

The conventional Crank-Nicolson finite-difference approximation to equation (1) at a typical point in the mesh may be written according to

$$\begin{aligned} \frac{u_{ij} - u_{ij}^*}{\Delta t} &= \frac{1}{2} \left\{ \bar{R}_{ij} h_\xi^{-2} \delta_\xi^2 + \bar{S}_{ij} h_\xi^{-1} h_\eta^{-1} \mu_\xi \delta_\xi \mu_\eta \delta_\eta + \bar{T}_{ij} h_\eta^{-2} \delta_\eta^2 \right. \\ &\quad \left. + \bar{P}_{ij} h_\eta^{-1} \mu_\eta \delta_\eta + \bar{Q}_{ij} h_\xi^{-1} \mu_\xi \delta_\xi + \bar{W}_{ij} \right\} \{u_{ij} + u_{ij}^*\} + \bar{I}_{ij}, \end{aligned} \quad (8)$$

where the asterisk denotes known values in the previous time plane and the overbar indicates a simple average between the current and the previous time plane (as, for example, in equation (6)). It is worthwhile to note that these difference approximations are second order accurate in both spatial directions and in time. Since the coefficients in equation (8) ( $R_{ij}$ ,  $S_{ij}$ ,  $T_{ij}$ , ...) depend in general on the dependent variable  $u$ , the difference equations are nonlinear and must be solved iteratively; for given estimates of the functional coefficients in equation (8), the system of equations is generally solved by systematically sweeping the spatial mesh using a Gauss-Seidel or an over-relaxation method. In the case when  $S \equiv 0$  in equation (1), the system of equation (1) may be written in the form

$$u_{ij} = \frac{1}{a_{ij}} \left\{ f_{ij} - b_{ij} u_{i+1,j} - c_{ij} u_{i-1,j} - d_{ij} u_{i,j+1} - e_{ij} u_{i,j-1} \right\}, \quad (9)$$

at a typical point in the mesh; for equations containing a second order cross derivative (and hence having a nonzero  $S$ ), the right side of equation (9) will also involve terms with  $u_{i+1,j+1}$ ,  $u_{i+1,j-1}$ ,  $u_{i-1,j+1}$  and  $u_{i-1,j-1}$ . In a point-by-point iterative sweep through the spatial mesh, current estimates of the values on the right side of equation (9) are used to define the new iterate for  $u_{ij}$  at each mesh point. Sweeping of the mesh is continued at each time step until convergence is obtained and this typically may require on the order of 10 to 50 iterations (ref. 1). Generally, convergence occurs

rapidly if the system of difference equations is diagonally dominant; this implies that the coefficient of the pivotal element must be greater or equal in magnitude than the sum of absolute values of the coefficients of the other elements in the difference equation. For the system defined by equation (9), diagonal dominance occurs if

$$|a_{ij}| \geq |b_{ij}| + |c_{ij}| + |d_{ij}| + |e_{ij}|, \quad (10)$$

for each point in the mesh.

#### Upwind-Downwind Differencing

It is well known that the central difference approximations (7c) and (7d) for the first derivative terms in equation (1) can lead to difference equations of the form (9) which are not diagonally dominant. In situations where  $P_{ij}$  and  $Q_{ij}$  are large (corresponding to strong local convection effects), the difference equations associated with equations (8) may fail to be diagonally dominant over a significant region of the flow field and the iterative sweeping of the mesh will fail to converge (refs. 1 and 2). In the problem studied by Doligalski and Walker (ref. 1) this phenomenon occurs as a viscous boundary-layer flow near a wall proceeds into a strong interaction with an outer effectively inviscid flow; in such cases, local strong updrafts occur in the boundary layer near the wall. Doligalski and Walker (ref. 1) have described an upwind-downwind differencing procedure which is second order accurate in both space and time and which always leads to a diagonally dominant set of difference equations (see also refs. 8 and 9). The procedure only affects the treatment of the terms  $P\partial u/\partial \eta$  and  $Q\partial \eta/\partial \xi$  in equation (1) and will be described briefly here to set the stage for the algorithm that will be described in the next section.

Consider first the term  $P\partial u/\partial \eta$ . In the conventional Crank-Nicolson method, each term in equation (1) is approximated at a point midway between the current and previous time planes; a simple average for all terms is then carried out along the averaging path labelled (a) in figure 2(a). However, it is easily demonstrated that an average along any path through the central point yields a second order accurate result. In particular consider the situation where  $\bar{P}_{ij} > 0$  at the typical mesh point and the averaging path labelled (b) in figure 2(a); this path intersects the current time plane at a point midway between the points  $(i, j+1)$  and  $(i, j)$ , and the previous time plane midway between the points  $(i, j)$  and  $(i, j-1)$ . Consequently central differences may be used for  $\partial u/\partial \eta$  in both the current and previous time planes. For  $\bar{P}_{ij} < 0$ , the averaging takes place along the line labelled (c) in figure 2(a). The difference approximations for  $P\partial u/\partial \eta$  then may be written according to

$$\bar{P}_{ij} \frac{\partial u}{\partial \eta} = \begin{cases} \frac{\bar{P}_{ij}}{2h_\eta} \left( u_{i,j+1} - u_{i,j} + u_{i,j}^* - u_{i,j-1}^* \right), & \bar{P}_{ij} \geq 0, \\ \frac{\bar{P}_{ij}}{2h_\eta} \left( u_{i,j} - u_{i,j-1} + u_{i,j+1}^* - u_{i,j}^* \right), & \bar{P}_{ij} < 0. \end{cases} \quad (11)$$

It is easily confirmed that these approximations always act to enhance the diagonal dominance of the difference equations; in addition, the approximations are second order accurate in both  $h_\eta$  and  $\Delta t$ . A similar approach may be adopted for  $Q \partial u / \partial \xi$  and the averaging paths in this case are indicated schematically in figure 2(b); here the average for  $\partial u / \partial \xi$  takes place along either path (d) or path (e) depending on the sign of  $\bar{Q}_{ij}$ . The difference approximations are

$$\bar{Q}_{ij} \frac{\partial \bar{u}}{\partial \xi} = \begin{cases} \frac{\bar{Q}_{ij}}{2h_\xi} (u_{i+1,j} - u_{i,j} + u_{i,j}^* - u_{i-1,j}^*), & \bar{Q}_{ij} \geq 0, \\ \frac{\bar{Q}_{ij}}{2h_\xi} (u_{i,j} - u_{i-1,j} + u_{i+1,j}^* - u_{i,j}^*), & \bar{Q}_{ij} < 0, \end{cases} \quad (12)$$

which are second order accurate in  $h_\xi$  and  $\Delta t$ .

It is convenient to write these difference approximations in a more compact form by first introducing the enlargement operators (ref. 7) in the  $\xi$  and  $\eta$  directions defined by

$$E_\xi^{1/2} u(\xi, \eta) = u(\xi + h_\xi/2, \eta), \quad E_\eta^{1/2} u(\xi, \eta) = u(\xi, \eta + h_\eta/2), \quad (13)$$

respectively. Now define operators  $x_\eta^+$  and  $x_\eta^-$  according to

$$x_\eta^+ = \delta_\eta E_\eta^{\text{sgn}(\bar{P}_{ij}/2)} \quad (14a)$$

$$x_\eta^- = \delta_\eta E_\eta^{-\text{sgn}(\bar{P}_{ij}/2)} \quad (14b)$$

where

$$\text{sgn}(\bar{P}_{ij}) = \begin{cases} 1 & P_{ij} \geq 0, \\ -1 & P_{ij} \leq 0, \end{cases} \quad (15)$$

It follows that equation (11) may be written

$$\bar{P}_{ij} \frac{\partial \bar{u}}{\partial \eta} = \frac{\bar{P}_{ij}}{2h_\eta} \left\{ x_\eta^+ u_{ij} + x_\eta^- u_{ij}^* \right\}. \quad (16)$$

In a similar manner, define operators  $x_\xi^+$  and  $x_\xi^-$  according to



$$x_{\xi}^{+} = \delta_{\xi} E_{\xi}^{\text{sgn}(\bar{Q}_{ij}/2)} \quad (17a)$$

$$x_{\xi}^{-} = \delta_{\xi} E_{\xi}^{-\text{sgn}(\bar{Q}_{ij}/2)} \quad (17b)$$

and it follows that equation (12) may be written in the form

$$\bar{Q}_{ij} \frac{\partial u}{\partial \xi} = \frac{\bar{Q}_{ij}}{2h_{\xi}} \left\{ x_{\xi}^{+} u_{ij} + x_{\xi}^{-} u_{ij}^{*} \right\}. \quad (18)$$

The present algorithm will now be considered.

#### An Upwind-Downwind ADI Method

Suppose now that the conventional Crank-Nicolson approach is used for all terms in equation (1), except  $P\partial u/\partial \eta$  and  $Q\partial u/\partial \xi$ , where the averaging of the spatial partial derivatives takes place along the line labelled (a) in figure 2(a); the first derivative terms are approximated using equations (16) and (18). The resulting finite difference approximation to equation (1) may be written

$$\begin{aligned} u_{ij} - \frac{\Delta t}{2} \left\{ \bar{R}_{ij} h_{\xi}^{-2} \delta_{\xi}^2 + \bar{S}_{ij} h_{\xi}^{-1} h_{\eta}^{-1} \mu_{\xi} \delta_{\xi} \mu_{\eta} \delta_{\eta} + \bar{T}_{ij} h_{\eta}^{-2} \delta_{\eta}^2 \right. \\ \left. + h_{\eta}^{-1} \bar{P}_{ij} x_{\eta}^{+} + h_{\xi}^{-1} \bar{Q}_{ij} x_{\xi}^{+} + \bar{W}_{ij} \right\} u_{ij} \\ = u_{ij}^{*} + \frac{\Delta t}{2} \left\{ \bar{R}_{ij} h_{\xi}^{-2} \delta_{\xi}^2 + \bar{S}_{ij} h_{\xi}^{-1} h_{\eta}^{-1} \mu_{\xi} \delta_{\xi} \mu_{\eta} \delta_{\eta} + \bar{T}_{ij} h_{\eta}^{-2} \delta_{\eta}^2 \right. \\ \left. + h_{\eta}^{-1} \bar{P}_{ij} x_{\eta}^{-} + h_{\xi}^{-1} \bar{Q}_{ij} x_{\xi}^{-} + \bar{W}_{ij} \right\} u_{ij}^{*} + \Delta t \bar{F}_{ij}. \end{aligned} \quad (19)$$

This equation defines a difference equation at the typical mesh point, which can be utilized to carry out a point-by-point sweep through the mesh at each time step. Experience with this algorithm (refs. 1 and 2), for the unsteady boundary-layer equations in Eulerian coordinates, indicates that the method produces results which are essentially the same as those produced with the conventional Crank-Nicolson method for well behaved unsteady boundary-layer flows. However, in situations where strong convective effects occur, the standard Crank-Nicolson method fails to converge (refs. 1 and 2) as the difference equations begin to lose the diagonal dominance property; in such cases the upwind-downwind differencing scheme continued to produce converged results at each time step and the method could be used well beyond the point in time where the conventional Crank-Nicolson method failed (refs. 1 and 2).

One disadvantage of the point-by-point iterative methods is that a significant number of iterations may be required to obtain convergence at each time step. An attractive alternative is an ADI algorithm of the type described by Beam and Warming (ref. 5). In this approach, the operators on the left side of equation (19) are factored into operators in the  $\xi$  and  $\eta$  directions. In the case of a second order cross derivative, the factorization is not easily accomplished and consequently the difference expressions resulting from this term are taken to the right side of the equation. A factored form of equation (19) may be obtained according to

$$\left\{ 1 - \frac{\alpha \Delta t}{h_\eta^2} \left( \bar{T}_{ij} \delta_\eta^2 + h_\eta \bar{P}_{ij} x_\eta^+ \right) \right\} \left\{ 1 - \frac{\alpha \Delta t}{h_\xi^2} \left( \bar{R}_{ij} \delta_\xi^2 + h_\xi \bar{Q}_{ij} x_\xi^+ \right) \right\} u_{ij} = D_{ij}, \quad (20)$$

where

$$\alpha = \frac{1}{2 - \Delta t \bar{W}_{ij}}, \quad (21)$$

$$\begin{aligned} D_{ij} = & 2\alpha u_{ij}^* + \alpha \Delta t \left( \bar{T}_{ij} h_\eta^{-2} \delta_\eta^2 + \bar{P}_{ij} h_\eta^{-1} x_\eta^- + \bar{R}_{ij} h_\xi^{-2} \delta_\xi^2 \right. \\ & \left. + \bar{Q}_{ij} h_\xi^{-1} x_\xi^- + \bar{W}_{ij} \right) u_{ij}^* \\ & + \alpha \Delta t \left( \bar{S}_{ij} h_\xi^{-1} h_\eta^{-1} \mu_\xi \delta_\xi \mu_\eta \delta_\eta \right) (u_{ij} + u_{ij}^*) + 2\alpha \Delta t \bar{T}_{ij}. \end{aligned} \quad (22)$$

Note that equations (19) and (20) are not completely equivalent and differ by terms  $O((\alpha \Delta t)^2)$ ; however, the temporal truncation error associated with the original Crank-Nicolson method is also  $O((\Delta t)^2)$  and thus equation (20) may be regarded as a second order accurate difference approximation for equation (1).

The set of difference equations given by equation (20) may be solved as a sequence of diagonally dominant tridiagonal matrix problems in the following manner. Let  $\tilde{u}_{ij}$  be an intermediate dependent variable defined by

$$\tilde{u}_{ij} - \frac{\alpha \Delta t}{h_\xi^2} \left\{ \bar{R}_{ij} \delta_\xi^2 + h_\xi \bar{Q}_{ij} x_\xi^+ \right\} \tilde{u}_{ij} = D_{ij}, \quad (23)$$

it then follows that equation (20) becomes

$$u_{ij} - \frac{\alpha \Delta t}{h_\eta^2} \left\{ \bar{T}_{ij} \delta_\eta^2 + h_\eta \bar{P}_{ij} x_\eta^+ \right\} u_{ij} = \tilde{u}_{ij}. \quad (24)$$

To initiate a solution for  $u_{ij}$  in the current time plane, equation (23) is first solved along all lines of constant  $\eta$  using a direct method of solution for triadiagonal matrix problems (for example, the Thomas algorithm); this sweep through the mesh defines current estimates of the intermediate variable  $\tilde{u}_{ij}$  at all internal spatial mesh points. Note that boundary conditions for  $\tilde{u}_{ij}$  along the maximum and minimum values of  $\eta$  for this sweep may be obtained using equation (24). In the second phase of the procedure, the computed values of  $\tilde{u}_{ij}$  are used on the right side of equation (23) which now defines a sequence of triadiagonal matrix problems for  $u_{ij}$  along lines of constant  $\xi$ . Again this sequence of problems is solved by a direct method for each value of  $\xi$  in the mesh. The net result of this procedure is an estimate for each  $u_{ij}$  at each internal mesh point. Note that another version of the algorithm is obtained by interchanging the operators in equations (23) and (24) and for which a set of triadiagonal problems is first solved along lines of constant  $\xi$ ; this would be followed by solving a set of triadiagonal problems along lines of constant  $\eta$ .

It should be noted that it is possible to apply this algorithm in a noniterative manner by selecting the time step to be small enough so that iteration is not necessary. However, in flow problems in which rapid changes begin to develop in the flow field, it is prudent to use a limited amount of iteration at each timestep. Iteration is indicated since the coefficients  $\bar{R}_{ij}$ ,  $\bar{Q}_{ij}$ ,  $\bar{T}_{ij}$ , and  $\bar{P}_{ij}$  in equations (23) and (24) are averages between the current and previous time planes and consequently are implicit functions of the dependent variable  $u$ . Furthermore, the right side of equation (23), which is defined in equation (22), contains values of  $u_{ij}$  explicitly, in terms associated with the second order cross derivative; consequently the value of  $D_{ij}$  at each mesh point should be recomputed and at least one more iteration carried out.

## DISCUSSION

In this study, an upwind-downwind ADI method has been developed. In ADI methods, the two-dimensional spatial difference operators are factored into two sequential sets of triadiagonal matrix problems, one in each of the coordinate directions. These types of methods have proved to be very efficient due to the fact that (1) direct solvers for triadiagonal matrix problems are relatively fast and accurate, and (2) the process of solving for a line of information at once appears to be more effective at communicating boundary information to internal mesh points as opposed to systematic point-by-point sweeping of the mesh.

Conventional ADI methods of the type given by Beam and Warming (ref. 5) may also be applied to equations like equation (1). Such methods employ standard central differences for the spatial derivatives and are generally used in a noniterative mode. The present method, as well as the explicit Beam and Warming (ref. 5) algorithm, has recently been used to compute the unsteady boundary-layer development due to a vortex convected above a wall (ref. 6); at a certain stage in this flow, the boundary layer begins to thicken very rapidly and strong convective effects develop locally. Indeed this is the type of boundary-layer flow which rapidly evolves into an viscous-inviscid interaction with the outer inviscid flow in the form of a boundary-layer

eruption. In the early stages of the motion, both methods were able to compute the flow evolution very efficiently. However, once strong convection effects began to develop locally, the Beam and Warming (ref. 7) algorithm was unable to successfully track the flow evolution; this failure is believed due to a lack of diagonal dominance which eventually leads to failure of the method.

The present method was also compared to the solution process of systematic point-by-point iterative sweeping of the mesh. Here again the comparison dramatically favored the present method for the example problem (ref. 6). Time steps ten times larger could be taken using the present scheme and at the same time only about one tenth the iterations were required at each time step. Finally, as the convective effects strengthened it proved impossible to get converged solution at all, using the point-by-point iteration.

Lastly, it is worthwhile to note that although the present method may be used in a noniterative manner, use of iteration at each time step is considered prudent and is recommended. In the unsteady boundary-layer problem studied in reference 6, a maximum number of iterations per time step was assigned a priori; if the maximum was exceeded for any given time step, the calculation was restarted from the previous time plane with a smaller time step. In this manner, the number of iterations per time step gives an indication of the need to reduce (or increase) the time step. Ideally the time step should be such that on the order of two or three iterations only are required each time step. If very little change occurs in the solution over each iteration, a larger time step can be used.

## APPENDIX

In this Appendix, the central difference operator notation used in this study is explicitly defined for completeness. Here  $(\xi, \eta)$  are independent spatial coordinates and the uniform mesh spacing in each direction is noted by  $h_\xi$  and  $h_\eta$  respectively. The enlargement operators in the  $\xi$  and  $\eta$  directions are defined by

$$E_\xi^p u(\xi, \eta) = u(\xi - ph_\xi, \eta), \quad (\text{A.1})$$

$$E_\eta^q u(\xi, \eta) = u(\xi, \eta + qh_\eta), \quad (\text{A.2})$$

where  $p$  and  $q$  are constants; these operators simply shift the appropriate argument of the function. The central difference operators  $\delta_\xi$  and  $\delta_\eta$  are defined by

$$\delta_\xi = E_\xi^{1/2} - E_\xi^{-1/2}, \quad \delta_\eta = E_\eta^{1/2} - E_\eta^{-1/2}, \quad (\text{A.3})$$

and consequently

$$\delta_\xi u(\xi, \eta) = u(\xi + h_\xi/2, \eta) - u(\xi - h_\xi/2, \eta), \quad (\text{A.4})$$

$$\delta_\eta u(\xi, \eta) = u(\xi, \eta + h_\eta/2) - u(\xi, \eta - h_\eta/2), \quad (\text{A.5})$$

corresponding to a central difference with one spatial variable held fixed. Finally the averaging operators  $\mu_\xi$  and  $\mu_\eta$  are defined by

$$\mu_\xi = \frac{1}{2} (E_\xi^{1/2} + E_\xi^{-1/2}), \quad (\text{A.6})$$

$$\mu_\eta = \frac{1}{2} (E_\eta^{1/2} + E_\eta^{-1/2}), \quad (\text{A.7})$$

and therefore

$$\mu_\xi u(\xi, \eta) = \frac{1}{2} \left\{ u(\xi + h_\xi/2, \eta) + u(\xi - h_\xi/2, \eta) \right\}, \quad (\text{A.8})$$

$$\mu_\eta u(\xi, \eta) = \frac{1}{2} \left\{ u(\xi, \eta + h_\eta/2) + u(\xi, \eta - h_\eta/2) \right\}, \quad (\text{A.9})$$

corresponding to an average with one spatial variable held fixed.

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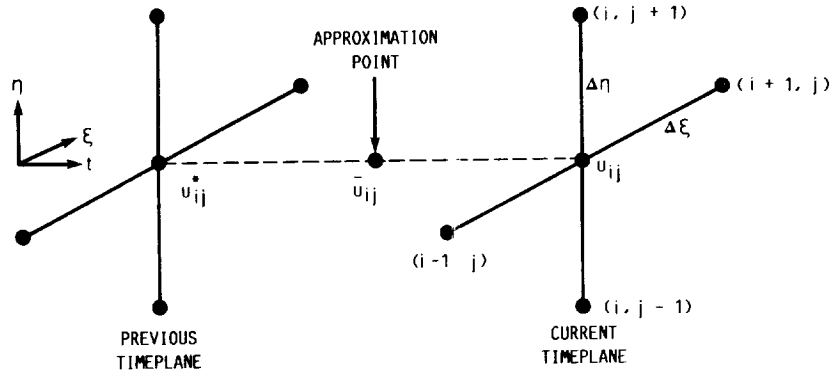


FIGURE 1. - GRID STRUCTURE AND NOTATION FOR CONVENTIONAL CRANK-NICOLSON METHOD.

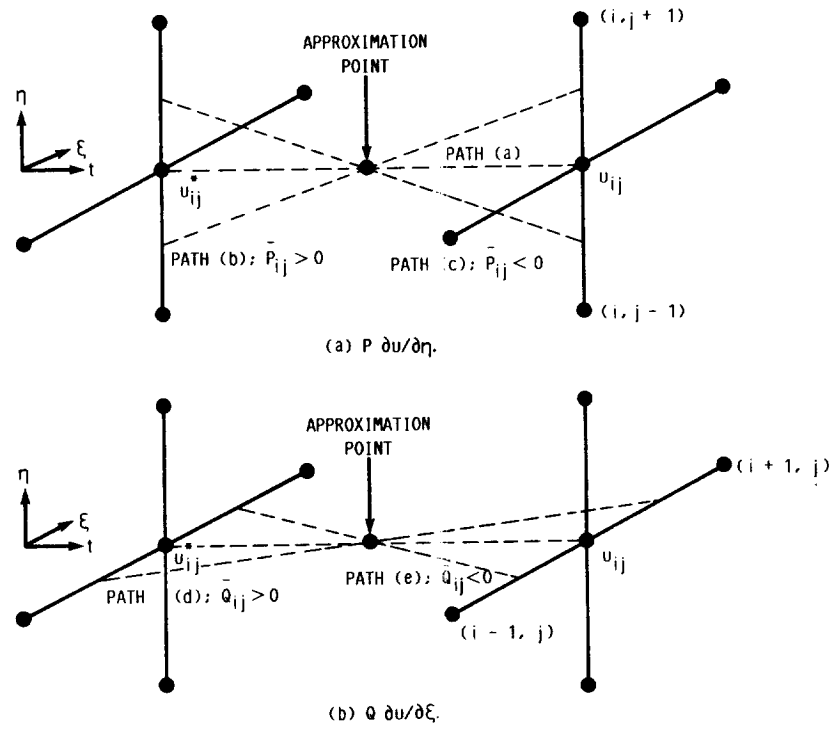


FIGURE 2. - AVERAGING PATH FOR CONVECTIVE DERIVATIVE TERMS.



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16. Abstract  An implicit ADI numerical method for the calculation of two-dimensional unsteady flows with strong convection effects is described. The method is based upon the conventional Crank-Nicholson approach for parabolic equations but an upwind-downwind differencing is used for the first order spatial derivatives associated with convection. The differencing is carried out in the current and previous time plane in such a way that the algorithm is second order accurate in both space and time. The difference equations are factored into sequential operators, one in each independent spatial variable; the solution at each time step may then be computed as a sequence of tridiagonal matrix problems. The method may be used in a noniterative manner although iteration at each time step is recommended in situations where the effects of convection are strong.					
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